Challenge Problems 2

Jacob Terkel

November 29, 2021

Problem Index

Problem	1	2	3	4	5	6	7
Difficulty/10	3.5	6	1	4	10	6	7.5
Category	LG	NT	GM	AL	CO	NT	CO

Key:

- NT: Number Theory
- AL: Algebra
- CO: Combinatorics
- GM: Geometry
- LG: Logic

1. Let $\exists!!(x, y), P(x) \wedge P(y)$ mean that there are two distinct values of x and y such that the predicate P holds. Express this using standard set notation, logic, and the existence and universal quantifiers (but not the uniqueness quantifier).

2. Consider the Diophantine equation $7x^3 + 3y^2 = z^2$. Prove that x, y, z are divisible by 7.

3. Determine the minimum number of lines required to partition a convex n-gon that lies in $\mathbb{R}^2$ into triangles.

4. Let $\mathbb{Z}_p[X]$ be the ring of polynomials with coefficients in $\mathbb{Z}_p$ for prime p. Determine the group structure of $(\mathbb{Z}_p[X]/(X^2 + X))$ (ie. find a more common group that this is isomorphic to.)

5. Let G be a finite Abelian group with order 9 or greater. Prove the following:

   Theorem 1. *$\mathbb{Z}_3^2$ is a subgroup of G if and only if there exists some $S \subseteq G$ where $|S| = 8$, such that one of the following holds for every 3-subset of S, A.*

   - $0 \in A$
   - $\{a, -a\} \subset A$, *and* $a \neq -a$.
   - $\{a, b, c\} = A$ *and* $a + b + c = 0$
   - $\{a, b, c\} = A$ *and* $a + b = c$ *(WLOG)*

   Feel free to ask for a hint on this one! It is far from easy!

6. Prove that there is no Gaussian integer α for which

$$|(\mathbb{Z}[i]/(\alpha))^\times| = 68$$

7. For a finite Abelian group G, let ρ_G be the minimum size of S_2 where S_2 is the set of all possible sums of two distinct elements in S, where S is some subset of G with cardinality 7. Prove that ρ_G is never more than 11.

   The use of a computer is highly encouraged for this problem!